Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(03)00093-5
J. Appl. Maths Mechs, Vol. 67, No. 4, pp. 553–564, 2003

0021-8928/$—see front matter

THE MOTION IN A PERFECT FLUID OF A BODY CONTAINING A MOVING POINT MASS†

V. V. KOZLOV and D. A. ONISHCHENKO

Moscow

e-mail: kozlo@pran.ru; onish@ipmnet.ru

(*Received* 25 *March* 2003)

The motion of a system (a rigid body, symmetrical about three mutually perpendicular planes, plus a point mass situated inside the body) in an unbounded volume of a perfect fluid, which executes vortex-free motion and is at rest at infinity, is considered. The motion of the body occurs due to displacement of the point mass with respect to the body. Two cases are investigated: (a) there are no external forces, and (b) the system moves in a uniform gravity field. An analytical investigation of the dynamic equations under conditions when the point performs a specified plane periodic motion inside the body showed that in case (a) the system can be displaced as far as desired from the initial position. In case (b) it is proved that, due to the permanent addition of energy of the corresponding relative motion of the point, the body may float upwards. On the other hand, if the velocity of relative motion of the point is limited, the body will sink. The results of numerical calculations, when the point mass performs random walks along the sides of a plane square grid rigidly connected with the body, are presented.

The classical problem of the motion of a rigid body in an infinite volume of a perfect fluid, which performs vortex-free motion and is at rest at infinity (see, for example, [1, 2]), allows of different generalizations, including to the case of a body of variable geometry. The free motion (when there are no external forces) of a variable body, under conditions when a change in the geometry of the masses of the body and its shape occurs due to the action of internal forces and is described by specified functions of the time with respect to a certain moving system of coordinates, was considered earlier in [3, 4]. In this formulation, the problem of the motion of variable body reduces to investigating the motion of such a system of coordinates. The following new effect was discovered [3, 4]: the law of variation of the geometry of the body can be chosen in such a way as to ensure displacement of the body to any point (as far away as desired) of the surrounding volume of fluid. Complete controllability of this system also turned out to be possible while preserving the shape of the external surface of the body (i.e. solely due to a change in the internal geometry of the masses). A unique condition is the fact that the connected masses of the body (which, we recall, depend only on the shape of its surface) should not all be equal to one another. Note that the results obtained previously on the possibility of unlimited motion of a variable body (see, for example, [5, 6]) are based on the use of mechanisms for controlling the geometry of the body for which the shape of its surface and its volume are changed. Below we investigate in more detail the mechanism of displacement of a body with a point mass inside it, and we also investigate the motion of this variable body in a uniform force field.

1. THE EQUATIONS OF MOTION

It is well known that the problem of the motion of a rigid body in an unbounded volume of a perfect fluid can be considered in a generalized formulation, when a change in the geometry of the body is allowed. Here we will investigate the case when the changeable body consists of a body (correspondingly a rigid body) and a point mass m, which is displaced inside it. We will assume that the motion of the whole system begins from a state of rest. The motion of the point with respect to the body is assumed to be specified in the sense that, in a system of coordinates rigidly connected with the body, the coordinates of the point are known functions of time. In this case the problem reduces to investigating the combined motion of the body (the container) in the fluid and of the point when there are time-dependent holonomic constraints. In accordance with the constraint elimination principle (see, for example, [2]), the motion of the composite body in a perfect fluid (the body plus fluid plus point system)

†*Prikl. Mat. Mekh.* Vol. 67, No. 4, pp. 620–633, 2003.

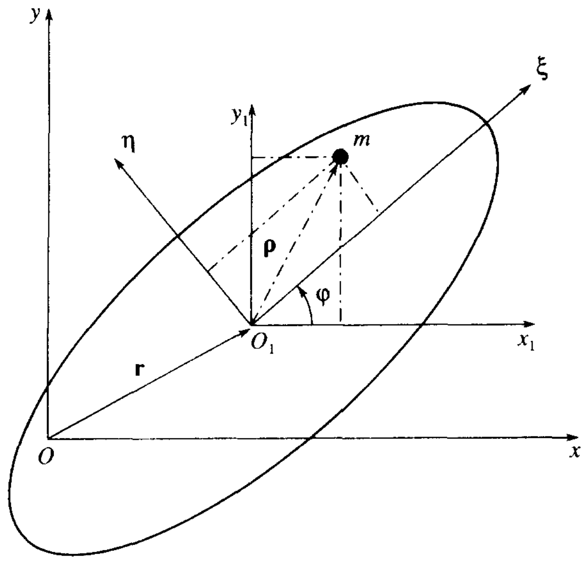

Fig. 1

can be interpreted as a classical problem of the motion of a rigid body in a fluid (a body plus fluid system) when acted upon by certain specified internal forces, generally time-dependent. These forces are obviously none other than reaction forces which arise as a result of the superposition of constraints which ensure the specified relative motion of the point in the body.

The problem can be investigated by different methods. Below we obtain a solution based on the use of the fundamental theorems of dynamics. We first introduce necessary notation, we obtain expressions for the fundamental dynamic characteristics of the mechanical system being investigated, and we derive the equations of motion. We will confine ourselves to the case when the body is symmetrical (both in shape and in mass distribution) about three mutually perpendicular planes, and performs plane motion. In this plane we fix a system of coordinates Oxy (see Fig. 1), relative to which the fluid is assumed to be at rest at infinity. We connect a moving system of coordinates $O_1\xi\eta$ with the centre of symmetry of the body O_1, this system of coordinates being oriented along the principle axes of inertia of the body and therefore occupying an unchanged position in it (we recall that the body itself is assumed to be absolutely rigid). Sometimes, for convenience, we will also use an auxiliary system of coordinates $O_1x_1y_1$, the axes of which remain parallel to the fixed axes Ox and Oy during motion.

As is well known [1], in the case of vortex-free motion of the fluid, the state of the body plus fluid system is uniquely defined by the position and velocity of the body. In the case of plane motion of the body (with corresponding plane motion of the point m) it is convenient to take as the generalized coordinates of the body plus fluid system the quantities x and y – the coordinates of the point O_1 and the angle φ between the O_1x_1 and $O_1\xi$ axes. We will characterize the position of the point m by the vector $\mathbf{r} + \rho$, where $\mathbf{r}$ is the radius vector of the point O_1 and ρ is the radius vector of the point m in the moving system of coordinates. Suppose x_1 and y_1 are projections of the vector ρ onto the x and y axes. We will assume the components ξ, η of the radius vector ρ in the moving system of coordinates $O_1\xi\eta$ to be specified functions of time. The relation between the quantities x_1, y_1 and $\xi(t)$, $\eta(t)$ is defined by the following formulae of rotation by the angle φ

$$x_1 = \xi(t)\cos\varphi - \eta(t)\sin\varphi, \quad y_1 = \xi(t)\sin\varphi + \eta(t)\cos\varphi \tag{1.1}$$

Note that relations (1.1) describe, in explicit form, two holonomic constraints imposed on the body plus fluid plus point system. Hence, the mechanical system considered is defined by five generalized coordinates x, y, φ, x_1 and y_1 when there are two holonomic constraints (1.1) and, consequently, has three degrees of freedom.

As usual, we will denote the differentiation of functions with respect to time by a dot. Then, the projections of the velocity vector of the point O_1 onto the fixed axes are equal to $\dot{x}$ and $\dot{y}$. If we denote the components of the same vector referred to the moving ξ and η axes by u and υ, then, like formulae (1.1), we will have

$$\dot{x} = u\cos\varphi - \upsilon\sin\varphi, \quad \dot{y} = u\sin\varphi + \upsilon\cos\varphi \tag{1.2}$$

and also the inverse relations

$$u = \dot{x}\cos\varphi + \dot{y}\sin\varphi, \quad \upsilon = -\dot{x}\sin\varphi + \dot{y}\cos\varphi$$

As is well known [1], in the (plane) case considered, the kinetic energy of the body plus fluid system can be represented in the form

$$T^{(0)} = \frac{1}{2}(a_1u^2 + a_2\upsilon^2 + b\omega^2)$$

Here the projections of the momentum $\mathbf{P}^{(0)}$ onto the moving axes are a_1u and $a_2\upsilon$, while the angular momentum $K_{O_1}^{(0)}$ about the point O_1 is equal to $b\omega$, where a_1, a_2 and b are the moments of inertia of the body with the connected masses (ignoring the point m) and $\omega = \dot{\varphi}$.

We will write expressions for the projections of the momentum vector $\mathbf{P}^{(0)}$ onto the fixed axes

$$P_x^{(0)} = a_1u\cos\varphi - a_2\upsilon\sin\varphi, \quad P_y^{(0)} = a_1u\sin\varphi + a_2\upsilon\cos\varphi \tag{1.3}$$

The total momentum $\mathbf{P}$ of the body plus fluid plus point system is equal to $\mathbf{P}^{(0)} + \mathbf{P}^{(m)} = \mathbf{P}^{(0)} + m\mathbf{V}^{(m)}$. Since the velocity vector of the point m has the components

$$\mathbf{V}^{(m)} = (\dot{x} + \dot{x}_1, \dot{y} + \dot{y}_1)_{xy} = (u - \omega\eta + \dot{\xi}, \upsilon + \omega\xi + \dot{\eta})_{\xi\eta}$$

the total momentum $\mathbf{P} = (P_\xi, P_\eta)$ can be written, in terms of projections onto the moving axes, in the form

$$\mathbf{P} = ((a_1 + m)u + m(\dot{\xi} - \omega\eta), (a_2 + m)\upsilon + m(\dot{\eta} + \omega\xi))_{\xi\eta} \tag{1.4}$$

We will now calculate the total kinetic moment of the system about the point O

$$K = (\mathbf{K}^{(0)} + \mathbf{K}^{(m)})_z = (\mathbf{K}_{O_1}^{(0)} + [\mathbf{r} \times \mathbf{P}^{(0)}] + m[(\mathbf{r} + \boldsymbol{\rho}) \times \mathbf{V}^{(m)}])_z \tag{1.5}$$

($(\cdot)_z$ is the projection of the vector onto the axis orthogonal to the x and y axes). Using the representation of the vectors in the Oxy axes used in expression (1.5), we obtain

$$K = b\omega + x[P_y^{(0)} + m(\dot{y} + \dot{y}_1)] - y[P_x^{(0)} + m(\dot{x} + \dot{x}_1)] + m[x_1(\dot{y} + \dot{y}_1) - y_1(\dot{x} + \dot{x}_1)]$$

Noting that

$$m(\dot{x} + \dot{x}_1) = P_x - P_x^{(0)}, \quad m(\dot{y} + \dot{y}_1) = P_y - P_y^{(0)}$$

and using relation (1.3), we finally obtain

$$K = b\omega + (x + x_1)P_y - (y + y_1)P_x + a_1u\eta - a_2\upsilon\xi \tag{1.6}$$

Obviously, the holonomic constraints (1.1) imposed on the system allow of a parallel transfer of the body plus fluid plus point system as a solid whole along the moving axes and rotation about the fixed Oz axis. By the fundamental theorems of dynamics [2], we have the relations

$$\frac{dP_x}{dt} = F_x, \quad \frac{dP_y}{dt} = F_y, \quad \frac{dK}{dt} = M \tag{1.7}$$

Where F_x and F_y are the projections of the sum of the external forces onto the fixed Ox and Oy axes respectively, while M is the total moment of the external forces about the point O. Relations (1.7) define, in the general case, a system of three second-order ordinary differential equations, which describe the plane motion of the body plus fluid plus point system for specified relative motion of the point and for specified external forces.

2. MOTION WHEN THERE ARE NO EXTERNAL FORCES

It was shown in [3] that when there are no external forces and the motion of the system begins from a state of rest, the problem is integrable. This integration is carried out in explicit form below.

Thus, $F_x = F_y = M = 0$ when there are no external forces. Hence, taking into account the fact that at the initial instant the system is at rest, we conclude from Eqs (1.7) that $P_x = P_y = 0$ (and of course $P_\xi = P_\eta = 0$), and $K = 0$. Finally, from relations (1.4) and (1.6) we obtain the relations

$$\begin{aligned} &(a_1+m)u+m(\dot{\xi}-\eta\omega) = 0, \quad (a_2+m)\upsilon+m(\dot{\eta}+\xi\omega) = 0 \\ &b\omega+a_1u\eta-a_2\upsilon\xi = 0 \end{aligned} \tag{2.1}$$

We will introduce the notation

$$\tilde{a}_1 = \frac{m}{a_1+m}, \quad \tilde{a}_2 = \frac{m}{a_2+m}$$

Expression u and υ from the first two relations (2.1)

$$u = \tilde{a}_1(\eta\omega-\dot{\xi}), \quad \upsilon = -\tilde{a}_2(\xi\omega+\dot{\eta}) \tag{2.2}$$

and substituting these expressions into the third, we obtain

$$\omega \equiv \dot{\varphi} = \frac{f_1}{f_2}; \quad f_1 = a_1\tilde{a}_2\eta\dot{\xi}-a_2\tilde{a}_1\xi\dot{\eta}, \quad f_2 = b+a_2\tilde{a}_2\xi^2+a_1\tilde{a}_1\eta^2 \tag{2.3}$$

The right-hand side of relation (2.3) is a specified function of time, and hence the notation $\varphi(t)$ of the moving frame of reference is found by straightforward integration

$$\varphi(t) = \varphi_0+\int_0^t\frac{f_1}{f_2}dt \tag{2.4}$$

Now, taking relations (2.2) into account, the right-hand sides of relations (1.2) become known functions of time and, consequently, the coordinates x and y of the point O_1 are found by straightforward integration.

We will represent these relations in a somewhat different form, which is more convenient for numerical integration. We will first determine the law of motion of the point O in the moving system of coordinates $O_1\xi\eta$. We will denote the components of the vector $\mathbf{r}_O = -\mathbf{r}$ in the moving axes by ξ_O and η_O. The theorem of the addition of velocities gives

$$\mathbf{V}_O \equiv \mathbf{0} = \mathbf{V}_{O_1}+[\omega\times\mathbf{r}_O]+(\mathbf{V}_O)_{\text{rel}} \tag{2.5}$$

Since, in the moving system of coordinate $O_1\xi\eta$

$$\mathbf{V}_{O_1} = (u, \upsilon), \quad [\omega\times\mathbf{r}_O] = (-\omega\eta_O, \omega\xi_O), \quad (\mathbf{V}_O)_{\text{rel}} = (\dot{\xi}_O, \dot{\eta}_O)$$

it follows from equalities (2.5) that

$$u-\omega\eta_O+\dot{\xi}_O = 0, \quad \upsilon+\omega\xi_O+\dot{\eta}_O = 0$$

Hence, taking relations (2.2) into account, we obtain

$$\dot{\xi}_O = \omega(\eta_O-\tilde{a}_1\eta)+\tilde{a}_1\dot{\xi}, \quad \dot{\eta}_O = \omega(-\xi_O+\tilde{a}_2\xi)+\tilde{a}_2\dot{\eta}$$

making the replacement

$$z_1 = \tilde{a}_2\xi-\xi_O, \quad z_2 = \tilde{a}_1\eta-\eta_O \tag{2.6}$$

we arrive at the system

$$\dot{z}_1 = \omega z_2 - \kappa\dot{\xi}, \quad \dot{z}_2 = -\omega z_1 + \kappa\dot{\eta}; \quad \kappa = \tilde{a}_1 - \tilde{a}_2 = \frac{m(a_2 - a_1)}{(a_1 + m)(a_2 + m)} \tag{2.7}$$

For convenience we will introduce the following set of complex variables

$$Z = z_1 + iz_2, \quad R = x + iy, \quad R_O = \xi_O + i\eta_O$$

and also two complex-valued functions of time, characterizing the relative motion of the point

$$\rho(t) = -\xi(t) + i\eta(t), \quad \sigma(t) = \tilde{a}_2\xi(t) + i\tilde{a}_1\eta(t)$$

In the new notation relations (2.6) take the form

$$Z = \sigma - R_O \tag{2.8}$$

while system (2.7) can be written as a linear first-order inhomogeneous equation

$$\dot{Z} = -i\omega(t)Z + \kappa\dot{\rho}(t) \tag{2.9}$$

The solution of the equation obtained has the form

$$Z(t) = S(t)e^{-i\varphi(t)} \tag{2.10}$$

The coefficient $S(t) = s_1(t) + is_2(t)$ is found from the relation

$$\dot{S}(t) = \kappa\dot{\rho}(t)e^{i\varphi(t)} \tag{2.11}$$

i.e.

$$S(t) = S(0) + \int_0^t \kappa\dot{\rho}(t)e^{i\varphi(t)}dt \tag{2.12}$$

Assuming, without loss of generality, that the points O and O_1 at the initial instant of time coincide (i.e. $\xi_O(0) = \eta_O(0) = 0$) and that $\varphi(0) = \varphi_0$, we obtain

$$S(0) = (\tilde{a}_2\xi(0) + i\tilde{a}_1\eta(0))e^{i\varphi_0}$$

From Eqs (2.8) and (2.10) we obtain

$$R_O(t) = \sigma(t) - S(t)e^{-i\varphi(t)} \tag{2.13}$$

Relation (2.13) together with (2.12) describes the relative motion of the point O.

It is now easy to obtain the absolute motion of the point O. In fact, in the complex planes $O_1\xi\eta$ and Oxy, rotated by an angle φ with respect to one another, the complex numbers $-R_O = -(\xi_O + i\eta_O)$ and $R = x + iy$ respectively specify the same point. Hence, taking relations (2.8), (2.9) and (2.11) into account

$$R(t) = -R_Oe^{i\varphi(t)} = S(t) - \sigma(t)e^{i\varphi(t)} \tag{2.14}$$

Relation (2.14) together with (2.4) describes the motion of the body in the fixed system of coordinates. No difficulties arise in carrying out numerical calculations using these relations. Below we will present some examples of the trajectories of the body in the fluid in the case of different relative motions of the point.

Following the approach described previously in [3], we will consider the problem of the possibility of unlimited motion of the casing of the body when there is periodic relative displacement of the point m. Suppose the functions $\xi(t)$ and $\eta(t)$ have period $T = 2\pi/\Omega$. In this case, by relations (2.3), the function φ is also periodic with the same period, and consequently

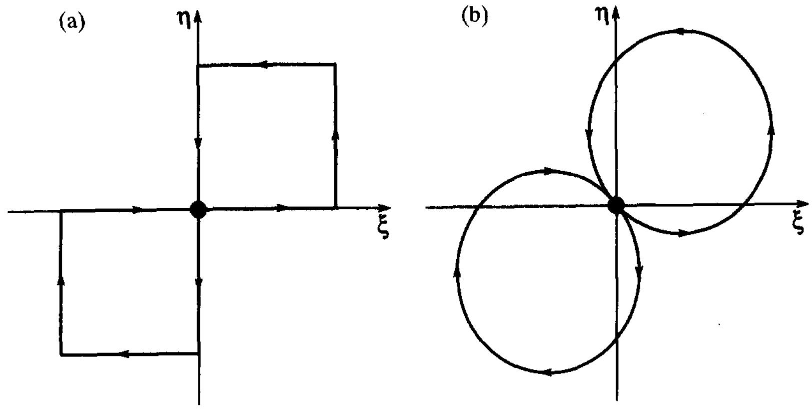

Fig. 2

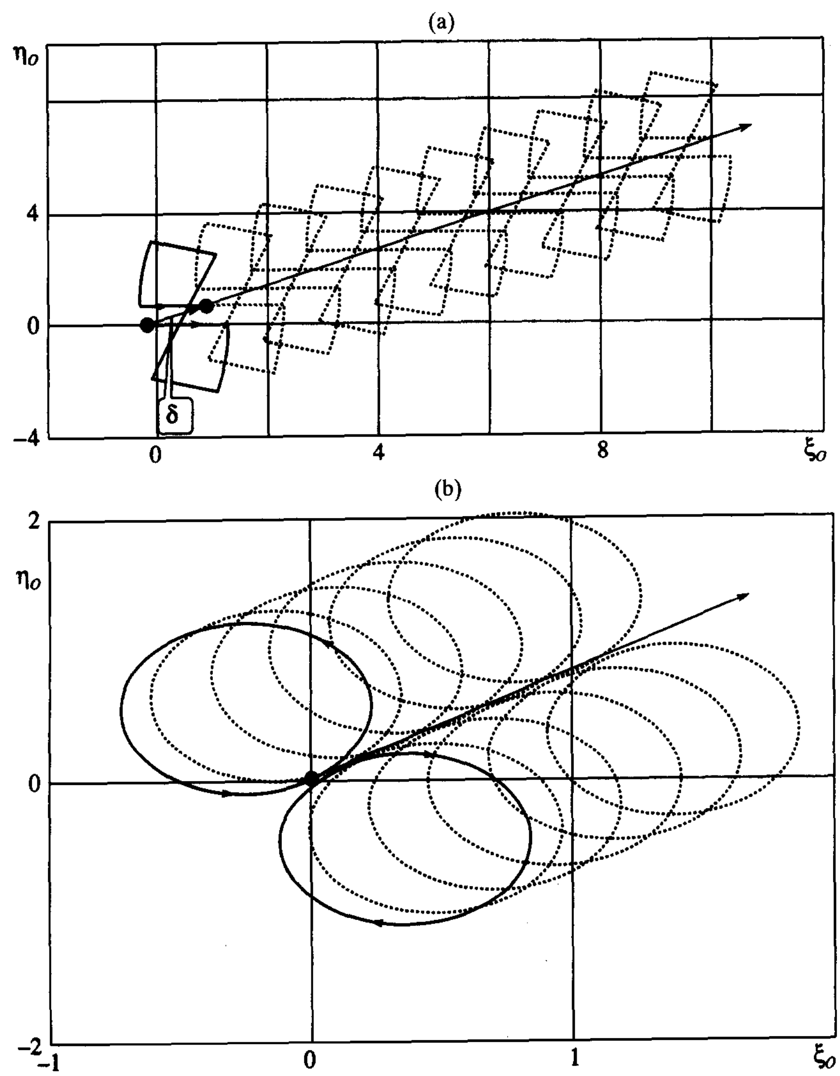

Fig. 3

$$\varphi(t) = \Lambda t + \Phi(t)$$

where Λ is the average value over a period of the function f_1/f_2, defined by relations (2.3), while $\Phi(t)$ is a certain function of period T. It follows from Eq. (2.11) that $\dot{S} = W(t)e^{i\Lambda t}$, where the function W is periodic with the same period. Using the expansion of W in a Fourier series, we arrive at the relation

$$\dot{S} = \sum_{n=-\infty}^{\infty} W_n e^{i(n\Omega+\Lambda)t}$$

When this equation is integrated in the case when there is resonance $n\Omega + \Lambda = 0$ in the representation of the function S, a non-zero secular term $W_n t$ generally appears, which in view of the boundedness of the term $\sigma(t)e^{i\varphi(t)}$ in expression (2.14) causes an unlimited departure of the body from the initial (fixed) position. Note that the simplest version of resonance is the case $\Lambda = 0$.

If there are no resonances (i.e. $n\Omega + \Lambda \neq 0$ for all n), the function S and, correspondingly, R turn out to be unbounded, and the body will therefore perform motion in a bounded neighbourhood of the origin of coordinates.

We will supplement the above analysis with characteristic representations of the trajectories of the body, constructed from the results of numerical integration of the governing system of differential equations. We will first illustrate the resonance case. It was shown in [4], that the self-intersecting contour in Fig. 2(a) is a resonance contour: when the point m moves along it with constant velocity the condition $\Lambda = 0$ is satisfied, and the integral over a period on the right-hand side of relation (2.12) is non-zero. In Fig. 3(a) we show the corresponding trajectory of the "motion" of the point O in the moving axes $O_1\xi\eta$ (in Fig. 3, for clarity, we have plotted values of $-\eta_O$). The displacement of the body over a period is equal to $\delta = W_0 T \neq 0$ and the body as a whole is displaced along a straight line, moving with constant velocity from the initial position. The direction of this line depends on the parameters of the body.

The trajectory shown in Fig. 3(b) corresponds to a resonance contour of a continuous figure-of-eight type, shown in Fig. 2(b), along which the point m moves with constant velocity. The pattern of motion of the system does not change qualitatively. This case is interesting in the fact that, unlike the contour with corners, here the functions $\xi(t)$, $\eta(t)$ are continuous. In Fig. 4 we show, for comparison, graphs of the variation with time of the generalized coordinates of the body ξ_O, $-\eta_O$, φ when the point moves along a contour of the figure-of-eight type in the resonant case (Fig. 4a) and in the non-resonant case (Fig. 4b). The latter version is obtained by a direct reduction in the radius of one of the circles of the "figure-of-eight". On can clearly see the difference in the nature of the variation with time of the function $\varphi(t)$: in the resonant case the body performs oscillatory motion while in the non-resonant case it performs rotational motion.

The trajectory of motion of the body in the non-resonant case is shown in Fig. 5. Note that, as one approaches the resonance condition – in this case as the radii of the two circles of the "figure-of-eight" approach another – the radius of curvature of the "frame" of the trajectory, along which the centre of the body O moves with oscillations of relatively small amplitude, increases without limit. As a consequence of this, in the corresponding finite time intervals, the trajectories of motion of the body in the resonant case (Fig. 3b) and in the non-resonant case (Fig. 5) become practically indistinguishable from one another.

3. ACCELERATED MOTIONS OF THE BODY

We draw attention to one important feature of the system being investigated in the case considered above, when no external forces act on the body and when the motion begins from a state of rest. It turns out that the trajectory of motion of the body is uniquely defined solely by the form of the trajectory of relative motion of the point m and is independent of the velocity of motion along this path.

The corresponding assertion can be formulated more accurately as follows. Suppose we know the motion of the system $\varphi(t)$, $R(t)$, when the relative trajectory of the point m is specified by the functions $\xi(t)$, $\eta(t)$, defined for $0 \leqslant t \leqslant t_*$. Then, if the point traverses the same trajectory, but with a different velocity (i.e. the motion of the point is described by the functions $\xi(h(t))$, $\eta(h(t))$, where the function $\tau \to h(t)$ is monotonic), the motion of the system is described by the functions $\varphi(h(t))$, $R(h(t))$.

We note further the following qualitative feature of the motion of the body in the resonant case. We define the "effective" velocity of the body in the case of the motion of a point m with constant velocity

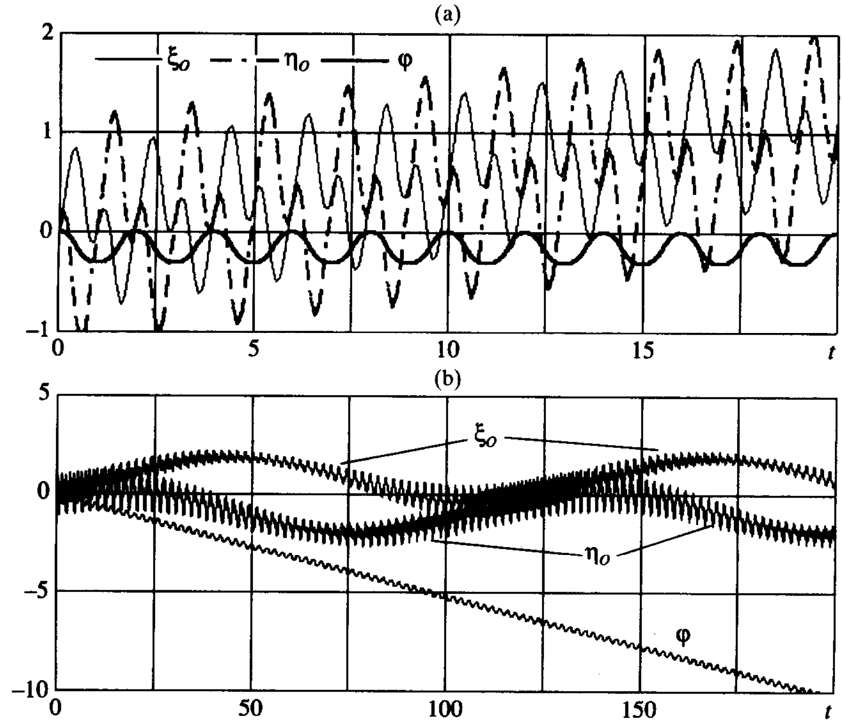
(a)
2
1
0
−1
0
5
10
15
t
ξ_O
η_O
φ
(b)
5
0
−5
−10
0
50
100
150
t
ξ_O
η_O
φ

Fig. 4

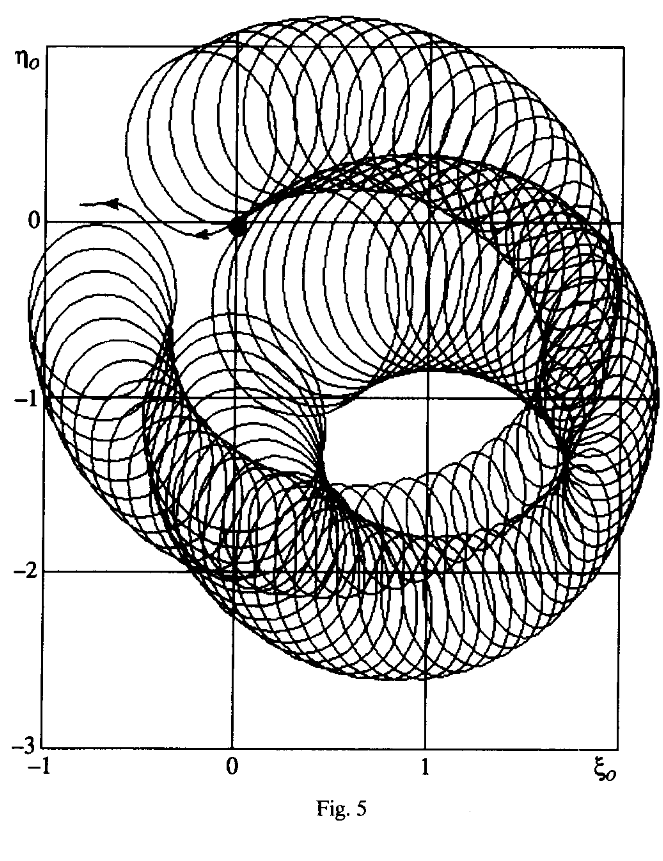
η_O
0
−1
-2
−3
−1
0
1
ξ_O

Fig. 5

along a closed contour γ as $\upsilon_e(t) = R(t)/t$. If when $\Lambda = 0$ the displacement of the body over a period T is equal to

$$\delta_0 = R(T) - R(0) \neq 0$$

it is then obvious that as $t \to \infty$

$$\upsilon_e(t) = \frac{\delta_0}{T} + \frac{O(t)}{t}$$

Hence, the body will be displaced on average with a velocity δ_0/T.

As follows directly from the above, when the point m moves over the same contour γ with a high velocity, the body obtains the same displacement δ_0, but this occurs after a shorter time. Hence, it immediately follows that by forcing the point to traverse the same "resonance" contour in an ever-decreasing time interval one can achieve accelerated motion of the body as a whole.

For example, consider the following law of motion of the point. Suppose the point moves with constant velocity over a closed resonance contour and performs one circuit in a time T. The point then performs two circuits at twice the velocity etc. In the nth cycle the point perform n circuits with a velocity exceeding the initial velocity by a factor of n.

Let us calculate the effective acceleration of the body. In a time $t_n = nT$ the displacement of the body (along a straight line) is equal to

$$\delta_n = (n+1)n\delta_0/2$$

The effective velocity of the body in the nth cycle is

$$\upsilon_n = n\delta_0/T = (\delta_0/T^2)t_n$$

Assuming, on this basis, that the motion of the body is uniformly accelerated with an effective acceleration $a_e = \delta_0/T^2$, we obtain the corresponding displacement after a time t_n: $\delta = a_e t_n^2/2 = n^2\delta_0/2$, which, for sufficiently large n, is identical with the quantity δ_n, thereby confirming the assumption of the accelerated nature of the motion of the body in the case considered.

We emphasise that the actual "tractive" force is produced as a result of the unlimited increase in the energy of relative motion of the point m.

4. MOTION IN A UNIFORM FORCE FIELD

We will complicate the problem and assume that the body plus fluid point system is in a gravitational field. Suppose F is a force, constant in value and direction, acting on the system considered; we will assume that this force is in the opposite direction of the y axis.

As follows from the results of Section 3, as a result of the proper motion of the point m inside the body under the action of internal forces, the body may constantly rise upwards. In this case the energy of relative motion of the point m increases with time. On the other hand, it turns out that if the relative velocity of the point m is limited (i.e. the functions $\dot{\xi}(t)$ and $\dot{\eta}(t)$ are bounded), then $y(t) \to \infty$ as $t \to \infty$. Hence, the body will finally fall downwards for as small a value of the force F as desired.

We will prove this assertion. The relations $\dot{P}_1 = 0$ and $\dot{P}_2 = -F$ follow from Eqs (1.7). Slightly extending the problem and assuming that, in the initial state (before the motion of the point begins), the body had a certain momentum $(c_1, c_2)_{xy}$, we obtain that $P_1 = c_1$ and $P_2 = -Ft + c_2$, where c_1 and c_2 are constants. By shifting the time origin one can always arrange that $c_2 = 0$. We will put $c_1 = c$.

Considering the momentum in projections onto the moving axes, and taking expression (1.4) into account, we obtain

$$(a_1 + m)u + m(\dot{\xi} - \omega\eta) = c\cos\varphi - Ft\sin\varphi \tag{4.1}$$

$$(a_2 + m)\upsilon + m(\dot{\eta} + \omega\xi) = -c\sin\varphi - Ft\cos\varphi \tag{4.2}$$

The motion of the body along the vertical is of interest, so we will use the second of relations (1.2). Expressing u and υ from Eqs (4.1) and (4.2) we obtain

$$\dot{y} = -\frac{m}{a_1+m}(\dot{\xi}-\eta\dot{\varphi})\sin\varphi + \frac{c\cos\varphi\sin\varphi}{a_1+m} - \frac{m}{a_2+m}(\dot{\eta}+\xi\dot{\varphi})\cos\varphi -$$

$$-\frac{c\sin\varphi\cos\varphi}{a_2+m} - Ft\left[\frac{\sin^2\varphi}{a_1+m}+\frac{\cos^2\varphi}{a_2+m}\right] \tag{4.3}$$

The expression in square brackets is no less than λ, where

$$\lambda = \min\left(\frac{1}{a_1+m}, \frac{1}{a_2+m}\right)$$

Consequently, the integral of the last term on the right-hand side of relation (4.3) tends to $-\infty$ at least no slower than $\lambda Ft^2/2$.

By our assumption, the function ξ and $\dot{\eta}$ are bounded. Consequently, the sum of the remaining terms in the expression for $\dot{y}$, not containing $\dot{\varphi}$, is bounded. Hence, the integral of these terms in the interval $[0, t]$ increases no faster than a linear function of t, i.e. it is bounded by a certain function μt.

Further, we will estimate the integral from the sum of the terms containing $\dot{\varphi}$

$$-\int_0^t\left[\frac{m}{a_1+m}\eta\frac{d}{dt}(\cos\varphi)+\frac{m}{a_1+m}\xi\frac{d}{dt}(\sin\varphi)\right]dt =$$

$$= \left(\frac{m}{a_1+m}\eta\cos\varphi+\frac{m}{a_1+m}\xi\sin\varphi\right)\Bigg|_0^t + \int_0^t\left(\frac{m\cos\varphi}{a_1+m}\dot{\eta}+\frac{m\sin\varphi}{a_2+m}\dot{\xi}\right)dt \tag{4.4}$$

Since ξ and η are bounded (the point m is always inside the body) together with their derivatives $\dot{\xi}$ and $\dot{\eta}$ (the kinetic energy of relative motion is bounded), the sum (4.4) does not exceed $\nu_1 + \nu_2 t$, where ν_1 and ν_2 are certain constants.

Thus

$$y(t) \le \nu_1 + (\nu_2+\mu)t - \lambda Ft^2/2$$

It remains to note that the right-hand side of this inequality approaches $-\infty$ as $t \to \infty$.

5. THE MOTION OF A BODY GENERATED BY A RANDOM WALK OF AN INTERNAL POINT

The question of the nature of the motion of a body when there are no external forces and when the internal point is displaced along a trajectory of "general form" is still an open one. Primarily, the case of the motion of a point with limited velocity is of interest. To investigate the problem in a first approximation we will use a probability approach, when we consider a random trajectory as the trajectory of general form. As an illustration we will present the results of numerical calculations of the trajectories of motion of a body which is generated by random displacements of the point inside the body.

We will confine ourselves to the following formulation. Suppose the point m performs a random walk (see, for example, [7]) along the nodes of a square grid, oriented along the axes of a moving system of coordinates. We will assume that during one step the point is displaced from the current node to one of the neighbouring nodes, and during even steps the motion occurs along one of the coordinates axes, while in odd steps motion occurs along the other axis (with a probability of 0.5 in the positive or negative directions). The law of motion is assumed to be the same for all steps. In order to ensure that the point remains inside the body, we use a scheme with reflection when carrying out the calculations. We will confine the region of possible motion of the point m to a square with centre at the point O. After the point is incident on the side of the square its displacement outside the limits of the square is forbidden: in a step when the direction of motion of the point is perpendicular to the boundary, the point is displaced inside the square with probability 1.

In the calculations, the size of the square was assumed to be equal to 20 linear units (steps). As was shown above, the configuration of the trajectory of the body depends on the configuration of the

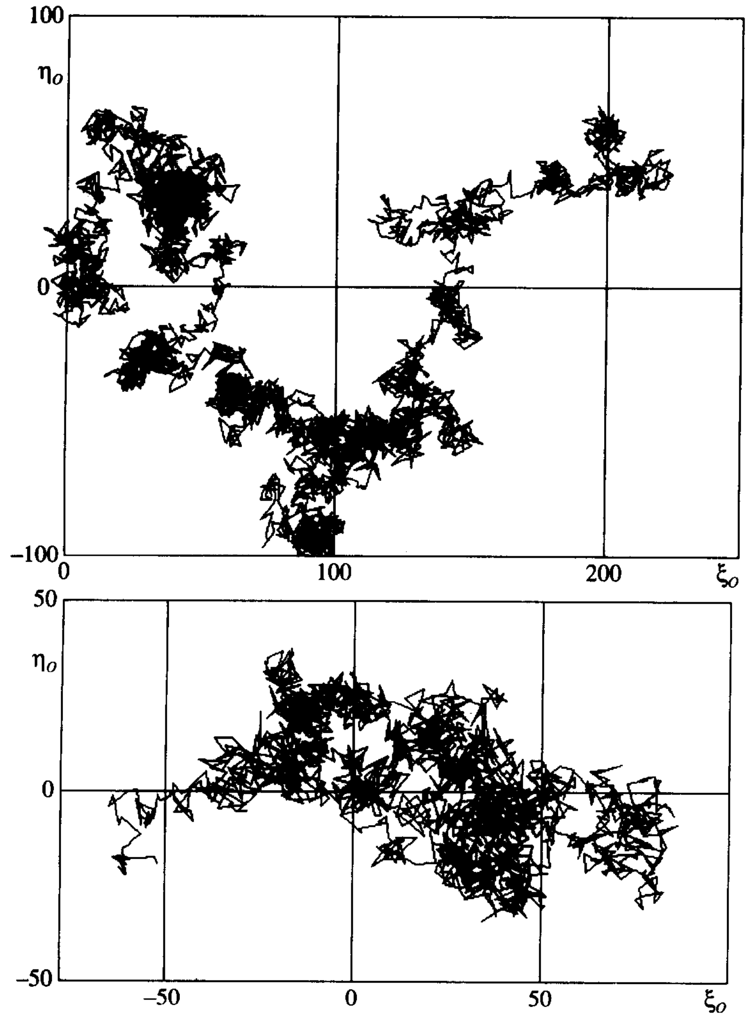

Fig. 6

trajectory of the point and does not depend on the "velocity regime" of motion of the point. For convenience in carrying out the procedure of numerical integration of this system, we chose the following form of motion of the point m: the point is at rest at nodes (this indicates that the whole body plus fluid plus point system is in a state of rest); the motion of the point from the initial node to the middle of the section, along which it moves, is a uniformly accelerated motion and then a uniformly decelerated motion.

This problem is described by the same relations (2.11) and (2.4) with the sole difference that in the numerical integration of them the values of the functions $\xi(t)$, $\eta(t)$, $\dot{\xi}(t)$ and $\dot{\eta}(t)$, which specify the relative motion of the point m, are calculated taking into account the random choice of the direction of motion. In the corresponding computer program we used a random number generator for these purposes.

In Fig. 6 we show two examples of the trajectories of motion of the body for the following values of the parameters: $a_1 = 1$, $a_2 = 7$, $b = 10$, and $m = 1$. In Fig. 7 we show corresponding graphs of the change with time of the distance from the centre of the body to the initial position.

In each calculation the point m made from 3×10^6 to 6×10^6 steps. The small number of calculations carried out – rather time consuming in fact – is too small a statistical sample to form a basis for any fundamental conclusions of a statistical nature. Nevertheless, we note the following qualitative results: first, when the duration of an "observation" (number of steps) is increased, it is found that there is an increase in the maximum distance that the body is displaced and, second, in the majority of cases the trajectory returned to the region of the origin of coordinates. Note that this behaviour corresponds to the properties of the classical random walk in a plane [7], the trajectories of which, as is well known, return to the initial point with probability 1.

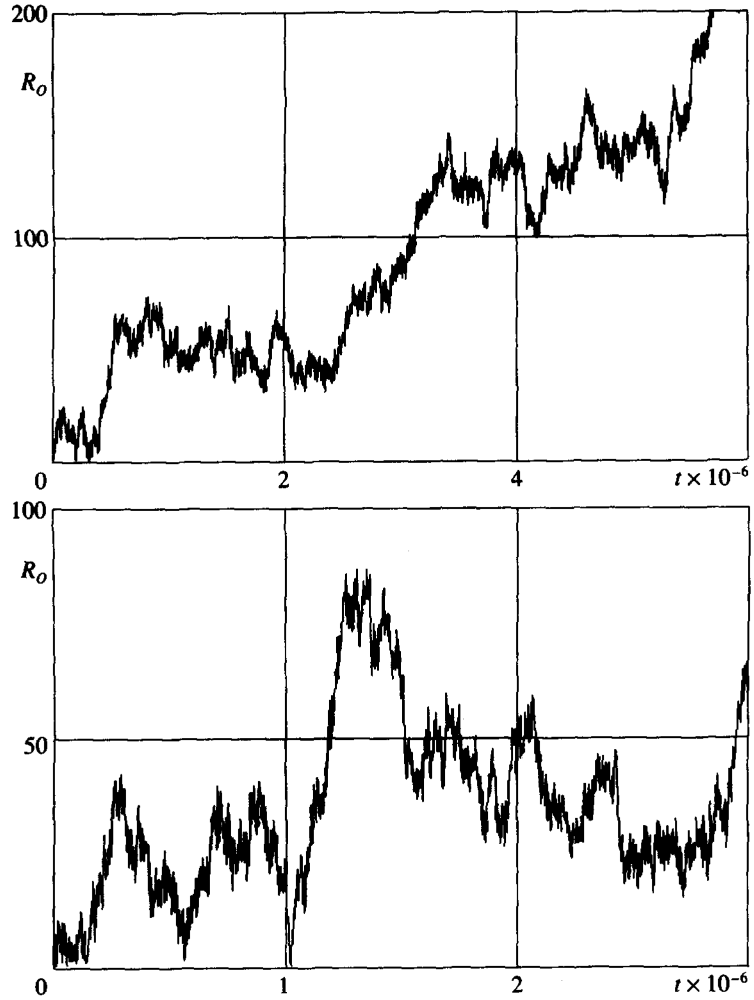

Fig. 7

This research was carried out within the framework of the "State Support of the Leading Scientific Schools" programme (NSh-136.2003.1) and the "Integration" programme (A-00-97).

REFERENCES

1. LAMB, H., *Hydrodynamic*. Dover Publ., New York; 1945.
2. KIRCHHOFF, G., *Vorlesungen über Mathematische Physik*. 1, *Mechanik*. B, Teubner, Leipzig, 1897.
3. KOZLOV, V. V. and RAMODANOV, S. M., The motion of a variable body in an ideal fluid. *Prikl. Mat. Mekh.*, 2001, **65**, 4, 592–601.
4. KOZLOV, V. V. and RAMODANOV, S. M., The motion in an ideal fluid of a body with a rigid shell and variable mass geometry. *Dokl. Ross. Akad. Nauk*, 2002, **382**, 4, 478–481.
5. MILOH, T. and GALPER, A., Self-propulsion of general deformable shapes in a perfect fluid. *Proc. Roy. Soc. London. Ser. A.*, 1993, **442**, 1915, 273–299.
6. GALPER, A. and MILOH, T., Dynamic equations of motion for a rigid or deformable body in an arbitrary non-uniform potential flow field. *J. Fluid Mech.*, 1995, **295**, 91–120.
7. FELLER, W., *An Introduction to Probability Theory and its Applications*. Wiley, New York, 1970

Translated by R.C.G